Linear Programming and The Simplex Method

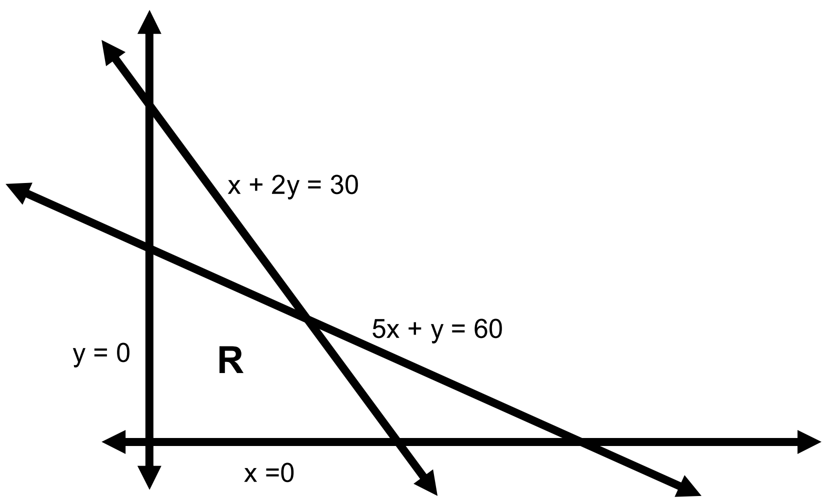
In calculus, we learned how to maximize or minimize a function of two functions using derivatives. We know turn our attention to maximizing linear functions using linear programming and the simplex method.

Linear Programming:
Linear programming is a technique in which we maximize or minimize a function using a shaded region from a graph.

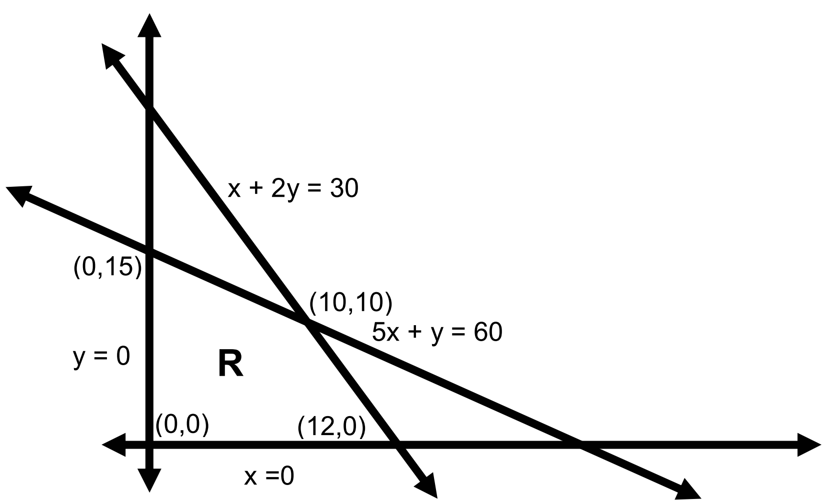
Example: Maximize $f = 3x + 2y$ subject to the following constraints:

(1) $x + 2y \leq 30$
(2) $5x + y \leq 60$
(3) $x \geq 0$, $y \geq 0$

1. Sketch all the constraints on the xy-coordinate axis. The region of interest is labeled as R.



2. Label all the vertices of the region R.



3. Test the vertices from the graph in f and the highest f value will yield the maximum.
   (0, 0) → f = 0
   (0, 15) → f = 30
   (10, 10) → f = 50 (MAXIMUM)
   (12, 0) → f = 36

So the maximum of f = 50 when x = 10, and y = 10.

<u>Simplex Method:</u>
The simplex method is a method which uses row row operations to obtain the maximum or minimum values of f.

Example: Maximize f = 3x + 2y subject to the following constraints:
   (1) $x + 2y \le 30$
   (2) $5x + y \le 60$
   (3) $x \ge 0$, $y \ge 0$

1. Set the f equation equal to zero by moving all the terms on the left hand side of the equation.
   $f = 3x + 2y \rightarrow -3x - 2y + f = 0$

2. Introduce slack variables (dummy variables) into the constraints.
   Since there are only two variables in the f equation, then we have two dummy variables, say $u \ge 0$ and $v \ge 0$.

$$x + 2y = 30 \rightarrow x + 2y + u = 30$$
$$5x + y = 60 \rightarrow 5x + y + v = 60$$

3. Set up the new system of equations with the dummy variables. Make sure the last equation is the f equation equal to zero.

$$\begin{aligned} x + 2y + u \quad &= 30 \\ 5x + y \quad + v &= 60 \\ -3x - 2y \quad + f &= 0 \end{aligned}$$

4. Write the system in augmented matrix form.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 30 \\ 5 & 1 & 0 & 1 & 0 & 60 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

5. Pick the largest negative number in the last row. This will be the pivot column.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 30 \\ 5 & 1 & 0 & 1 & 0 & 60 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

Largest (-) number

6. Take the numbers in the last column and divide by the numbers in the pivot column. The number that is the smallest will be the entry in the pivot column where the leading one occurs.

R1: 30/1 = 30
R2: 60/5 = 12 (SMALLEST)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 30 \\ 5 & 1 & 0 & 1 & 0 & 60 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot point

7. Make the pivot point in step 6 a one.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 30 \\ 5 & 1 & 0 & 1 & 0 & 60 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}R2} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 30 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

8. Make zeros above and/or below pivot point.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 30 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R3+3R1]{R1-R2} \left[\begin{array}{ccccc|c} 0 & 9/5 & 1 & -1/5 & 0 & 18 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ 0 & -7/5 & 0 & 3/5 & 1 & 36 \end{array}\right]$$

9. Check last row. If there are no negative numbers, then write the equations of the final matrix and set all dummy variables equal to zero. If not, repeat steps 5 – 8 again until there are no negative numbers in the last row.

--Pick largest negative number out of last row.

$$\left[\begin{array}{ccccc|c} 0 & 9/5 & 1 & -1/5 & 0 & 18 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ 0 & -7/5 & 0 & 3/5 & 1 & 36 \end{array}\right]$$

Largest (-) number

-- Take the numbers in the last column and divide by the numbers in the pivot column. The number that is the smallest will be the entry in the pivot column where the leading one occurs.

R1: 18/(9/5) = 10 (SMALLEST)
R2: 12/(1/5) = 60

$$\left[\begin{array}{ccccc|c} 0 & 9/5 & 1 & -1/5 & 0 & 18 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ 0 & -7/5 & 0 & 3/5 & 1 & 36 \end{array}\right]$$

Pivot point

-- Make the pivot point in step 6 a one.

$$\frac{5}{9}R1 \left[\begin{array}{ccccc|c} 0 & 1 & 5/9 & -1/9 & 0 & 10 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ 0 & -7/5 & 0 & 3/5 & 1 & 36 \end{array}\right]$$

-- Make zeros above and/or below pivot point.

$$\left[\begin{array}{cccccc|c} 0 & 1 & 5/9 & -1/9 & 0 & 10 \\ 1 & 1/5 & 0 & 1/5 & 0 & 12 \\ 0 & -7/5 & 0 & 3/5 & 1 & 36 \end{array}\right] \xrightarrow[R3+\frac{7}{5}R1]{R2-\frac{1}{5}R1} \left[\begin{array}{ccccc|c} 0 & 1 & 5/9 & -1/9 & 0 & 10 \\ 1 & 0 & -1/9 & 8/45 & 0 & 10 \\ 0 & 0 & 7/9 & 20/45 & 1 & 50 \end{array}\right]$$

--Since we have no more negative numbers in the last row, then we are finish. Writing the equations yields:

$$y + \frac{5}{9}u - \frac{1}{9}v = 10$$

$$x - \frac{1}{9}u - \frac{8}{45}v = 10$$

$$\frac{7}{9}u - \frac{20}{45}v + f = 50$$

--Set all the dummy variables equal to zero. This will yield the maximum value of the f equation and the max values of x and y.

$$y + \frac{5}{9}u - \frac{1}{9}v = 10 \rightarrow y = 10$$

$$x - \frac{1}{9}u - \frac{8}{45}v = 10 \rightarrow x = 10$$

$$\frac{7}{9}u - \frac{20}{45}v + f = 50 \rightarrow f = 50$$

So the maximum of f = 50 when x = 10, and y = 10 which is the same answer as before.