The Simplex Method: Step by Step with Tableaus

The simplex algorithm (minimization form) can be summarized by the following steps:

Step 0. Form a tableau corresponding to a basic feasible solution (BFS). For example, if we assume that the basic variables are (in order) $x_1, x_2, \ldots x_m$, the simplex tableau takes the initial form shown below:

$$
\begin{array}{cccccccccc}
x_1 & x_2 & \ldots & x_m & x_{m+1} & x_{m+2} & \ldots & x_j & \ldots & x_n & \text{RHS} \\
\hline
1 & 0 & \ldots & 0 & \bar{a}_{1,m+1} & \bar{a}_{1,m+2} & \ldots & \bar{a}_{1j} & \ldots & \bar{a}_{1n} & \bar{b}_1 \\
0 & 1 & \ldots & 0 & \bar{a}_{2,m+1} & \bar{a}_{2,m+2} & \ldots & \bar{a}_{2j} & \ldots & \bar{a}_{2n} & \bar{b}_2 \\
 & & & & & \vdots & & & & & \\
0 & 0 & \ldots & 0 & \bar{a}_{i,m+1} & \bar{a}_{i,m+2} & \ldots & \bar{a}_{ij} & \ldots & \bar{a}_{in} & \bar{b}_i \\
 & & & & & \vdots & & & & & \\
0 & 0 & \ldots & 1 & \bar{a}_{m,m+1} & \bar{a}_{m,m+2} & \ldots & \bar{a}_{mj} & \ldots & \bar{a}_{mn} & \bar{b}_m \\
\hline
0 & 0 & \ldots & 0 & \bar{c}_{m+1} & \bar{c}_{m+2} & \ldots & \bar{c}_j & \ldots & \bar{c}_n & (-z)
\end{array}
$$

Step 1. If each $\bar{c}_j \geq 0$, stop; the current basic feasible solution is optimal.

Step 2. Select q such that $\bar{c}_q < 0$ to determine which nonbasic variable is to become basic.

Step 3. Calculate the ratios $\bar{b}_i/\bar{a}_{iq}$ for $\bar{a}_{iq} > 0$, $i = 1, 2, \ldots, m$. If no $\bar{a}_{iq} > 0$, stop: the problem is unbounded. Otherwise, select p as the index i corresponding to the minimum ratio, i.e.,

$$\frac{\bar{b}_p}{\bar{a}_{pq}} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{iq}}, \bar{a}_{iq} > 0 \right\}$$

Step 4. Pivot on the pq-th element, updating all rows, including the z-row. Return to Step 1.

<u>Example 1</u>

The owner of a shop producing automobile trailers wishes to determine the best mix for his three products: flat-bed trailers, economy trailers, and luxury trailers. His shop is limited to working 24 days per month on metalworking and 60 days per month on woodworking for these products. The following table indicates the production data for the trailers.

	Flat-bed	Economy	Luxury	Resource Avail.
Metalworking (days)	0.5	2	1	24
Woodworking (days)	1	2	4	60
Unit Profit ($ H)	6	14	13	

Let the decision variables of the problem be:

$$
\begin{aligned}
x_1 &= \text{Number of flat-bed trailers produced per month} \\
x_2 &= \text{Number of economy trailers produced per month} \\
x_3 &= \text{Number of luxury trailers produced per month}
\end{aligned}
$$

The model is

$$
\begin{array}{lllllll}
\max & 6x_1 & + & 14x_2 & + & 13x_3 & \\
\text{s.t.} & & & & & & \\
& 0.5x_1 & + & 2x_2 & + & x_3 & \leq & 24 \\
& x_1 & + & 2x_2 & + & 4x_3 & \leq & 60 \\
& & & & & x & \geq & 0
\end{array}
$$

Let x_4 and x_5 be slack variables corresponding to unused hours of metalworking and woodworking capacity. Then the problem above is equivalent to the following minimization equation standard form problem.

$$
\begin{array}{llllllllll}
\max & 6x_1 & + & 14x_2 & + & 13x_3 & & & & \\
\text{s.t.} & & & & & & & & & \\
& 0.5x_1 & + & 2x_2 & + & x_3 & + & x_4 & & & = & 24 \\
& x_1 & + & 2x_2 & + & 4x_3 & + & & + & x_5 & = & 60 \\
& & & & & & & & & x & \geq & 0
\end{array}
$$

Obs: In standard form all variables are nonnegative and the RHS is also nonnegative.

The simplex method is performed step-by-step for this problem in the tableaus below. The pivot row and column are indicated by arrows; the pivot element is bolded. We use the greedy rule for selecting the entering variable, i.e., pick the variable with the most negative coefficient to enter the basis.

Tableau I

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	Ratio	Pivot
x_4	0.5	**2**	1	1	0	24	12	$\rightarrow$
x_5	1	2	4	0	1	60	30	
$(-z)$	-6	-14	-13	0	0	0		
Pivot		$\uparrow$						

Tableau II

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	Ratio	Pivot
x_2	0.25	1	0.5	0.5	0	12	24	
x_5	0.5	0	**3**	-1	1	36	12	$\rightarrow$
$(-z)$	-2.5	0	-6	7	0	168		
Pivot			$\uparrow$					

Tableau III

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	Ratio	Pivot
x_2	**1/6**	1	0	2/3	-1/6	6	36.0	$\rightarrow$
x_3	1/6	0	1	-1/3	1/3	12	72	
$(-z)$	-1.5	0	0	5	2	240		
Pivot	$\uparrow$							

Tableau IV

BASIS	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	6	0	4	-1	36
x_3	0	-1	1	-1	0.5	6
$(-z)$	0	9	0	11	0.5	294

Thus, the optimal value of the MINIMIZATION formulation is $z = -294$. This means that the optimal value of to the original MAXIMIZATION is $z = 294$. The optimal solution is $x = (36, 0, 6, 0, 0)$.

Comments

We do not have to change the objective from *max* to *min* in order to perform the simplex method. Suppose we'd like to keep the problem in maximization form. How must the steps outlined above be changed?

Step 0. SAME!

Step 1. If each $\bar{c}_j \leq 0$, stop; the current basic feasible solution is optimal.

Step 2. Select q such that $\bar{c}_q > 0$ to determine which nonbasic variable is to become basic.

Step 3. SAME!

Step 4. SAME!

That's it! Only the optimality test and the rule for determining variables eligible to enter the basis change. Everything else stays the same. Let's look at an example.

Example 2

Consider the problem

$$
\begin{array}{lcccccccc}
\max & 10x_1 & + & 12x_2 & + & 12x_3 & & \\
\text{s.t.} & & & & & & & \\
 & x_1 & + & 2x_2 & + & 2x_3 & \leq & 20 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 20 \\
 & 2x_1 & + & 2x_2 & + & x_3 & \leq & 20 \\
 & & & & & x & \geq & 0
\end{array}
$$

Let x_4, x_5, x_6 and be slack variables. Then the problem above is equivalent to

$$
\begin{array}{lcccccccccccc}
\max & 10x_1 & + & 12x_2 & + & 12x_3 & & & & & & & \\
\text{s.t.} & & & & & & & & & & & & \\
 & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & & = & 20 \\
 & 2x_1 & + & x_2 & + & 2x_3 & + & & + & x_5 & & = & 20 \\
 & 2x_1 & + & 2x_2 & + & x_3 & + & & & & + \; x_6 & = & 20 \\
 & & & & & & & & & & x & \geq & 0
\end{array}
$$

Tableau I

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Ratio	Pivot
x_4	1	**2**	2	1	0	0	20	10	$\rightarrow$
x_5	2	1	2	0	1	0	20	20	
x_6	2	2	1	0	0	1	20	10	
$(-z)$	10	12	12	0	0	0	0		
Pivot		$\uparrow$							

Obs: Either x_2 and x_3 could come into the basis. We've (somewhat) arbitrarily picked x_2. Given this choice, either x_4 or x_6 could leave the basis. Again, we've (somewhat) arbitrarily picked x_4 as the leaving variable.

Tableau II

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Ratio	Pivot
x_2	0.5	1	1	0.5	0	0	10	20	
x_5	1.5	0	1	-0.5	1	0	10	20/3	
x_6	**1**	0	-1	-1	0	1	0	0	$\rightarrow$
$(-z)$	4	0	0	-6	0	0	-120		
Pivot	$\uparrow$								

Tableau III

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Ratio	
x_2	0	1	1.5	1	0	-0.5	10	6.667	
x_5	0	0	**2.5**	1	1	-1.5	10	4.	$\rightarrow$
x_1	1	0	-1	-1	0	1	0		
$(-z)$	0	0	4	-2	0	-4	-120		
Pivot			$\uparrow$						

Tableau IV

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	0	0.4	-0.6	0.4	4
x_3	0	0	1	0.4	0.4	-0.6	4
x_1	1	0	0	-0.6	0.4	0.4	4
$(-z)$	0	0	0	-3.6	-1.6	-1.6	-136

So the optimal solution value is $z = 136$ and $x = (4, 4, 4, 0, 0, 0)$.