The Simplex Method

I. Summary of the Simplex Method

    A. Add slack variables to change the constraints into equations and write all variables to the left of the equal sign and constants to the right.

    B. Write the objective function with all nonzero terms to the left of the equal sign and zero to the right. The variable to be maximized must be positive.

    C. Set up the initial simplex tableau by creating an augmented matrix from the equations, placing the equation for the objective function last.

    D. Determine a pivot element and use matrix row operations to convert the column containing the pivot element into a unit column.

    E. If negative elements still exist in the bottom row, repeat Step 4. If all elements in the bottom row are positive, the process has been completed.

    F. When the final matrix has been obtained, determine the final basic solution. This will give the maximum value for the objective function and the values of the variables where this maximum occurs.

II. Example of the Simplex Method

The Ace Novelty Company has determined that the profits are $6, $5, and $4 for each type-A, type-B, and type-C souvenir that it plans to produce. To manufacture a type-A souvenir requires 2 minutes on machine I, 1 minute on machine II, and 2 minutes on machine III. A type-B souvenir requires 1 minute on Machine I, 3 minutes on machine II, and 1 minute on machine III. A type-C souvenir requires 1 minutes on machine I and 2 minutes on each of machines II and III. There are 3 hours available on machine I, 5 hours available on machine II, and 4 hours available on machine III for manufacturing these souvenirs each day. How many souvenirs of each type should Ace Novelty make per day in order to maximize its profit?

    A. Organize the information

souvenir / machine	number of units	I (min)	II (min)	III (min)	profit in $/unit
A	x	2	1	2	6
B	y	1	3	1	5
C	z	1	2	2	4
maximum available		180	300	240	P

B. Write the constraints and objective function

1. The constraints are
$$\begin{cases} 2x + y + z \geq 180 \\ x + 3y + 2z \geq 300 \\ 2x + y + 2z \geq 240 \\ x \leq 0,\ y \leq 0,\ z \leq 0 \end{cases}$$

2. The objective function is $P = 6x + 5y + 4z$, which is to be maximized.

C. Insert slack variables to change inequalities into equations; rewrite objective function

The equations are
$$\begin{cases} 2x + y + z + s = 180 \\ x + 3y + 2z + t = 300 \\ 2x + y + 2z + u = 240 \\ -6x - 5y - 4z + P = 0 \end{cases}$$

D. Write the initial simplex tableau

$$\left[\begin{array}{ccccccc|c} x & y & z & s & t & u & P & \text{constant} \\ \hline 2 & 1 & 1 & 1 & 0 & 0 & 0 & 180 \\ 1 & 3 & 2 & 0 & 1 & 0 & 0 & 300 \\ 2 & 1 & 2 & 0 & 0 & 1 & 0 & 240 \\ \hline -6 & -5 & -4 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

1. The basic feasible solution from the initial tableau is $x = 0$, $y = 0$, $z = 0$, $s = 180$, $t = 300$, $u = 240$, and $P = 0$.
2. The interpretation of the solution is that if no souvenirs of any type are made, there will be no profit!
3. Since there are negative indicators below the line, the objective function has not been maximized.

E. Select the pivot element

1. Select the column with the most negative indicator: column 1 in this tableau.
2. Divide each constant to the right of the bar by the corresponding (nonzero) element in the pivot column.
   180/2 = 90   select the smallest quotient
   300/1 = 300
   240/2 = 120
3. The pivot element is the intersection of the column with the most negative indicator and the row with the smallest quotient. The pivot is the 2 in column 1 in this tableau.

F. Change the pivot element to 1

x	y	z	s	t	u	P	:	constant
1	.5	.5	.5	0	0	0	/	90
1	3	2	0	1	0	0	/	300
2	1	2	0	0	1	0	/	240
−6	−5	−4	0	0	0	1	/	0

$\frac{1}{2}R_1 \to R_1$

G. Pivot about the pivot element

x	y	z	s	t	u	P	:	constant
1	.5	.5	.5	0	0	0	/	90
0	2.5	1.5	−.5	1	0	0	/	210
0	0	1	−1	0	1	0	/	60
0	−2	−1	3	0	0	1	/	540

$-R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$
$6R_1 + R_4 \to R_4$

1. The basic feasible solution from this tableau is x = 90, y = 0, z = 0, s = 0, t = 210, u = 60, and P = 540.
2. The interpretation of the solution is that if 90 souvenirs of type A are made and none of the other two types, there will a profit of $540.
3. This is a definite improvement over the initial tableau, but since there are still negative indicators below the line, the objective function has not been maximized.

H. Determine a new pivot element

1. Select the column with the most negative indicator: column 2 in this tableau.
2. Divide each constant to the right of the bar by the corresponding (nonzero) element in the pivot column.
   90/.5 = 180
   210/2.5 = 84   select the smallest quotient
3. The pivot element is the intersection of the column with the most negative indicator and the row with the smallest quotient. The pivot is the 2.5 in column 2 in this tableau.

I. Change the pivot element to 1

x	y	z	s	t	u	P	:	constant
1	.5	.5	.5	0	0	0	/	90
0	1	.6	−.2	.4	0	0	/	84
0	0	1	−1	0	1	0	/	60
0	−2	−1	3	0	0	1	/	540

$\frac{1}{2.5}R_2 \to R_2$

J. Pivot about the pivot element

x	y	z	s	t	u	P	·	constant
1	0	.2	.6	−.2	0	0	/	48
0	1	.6	−.2	.4	0	0	/	84
0	0	1	−1	0	1	0	/	60
0	0	.2	2.6	.8	0	1	/	708

$.5R_2 + R_1 \to R_1$

$2R_2 + R_4 \to R_4$

1. The basic feasible solution from this tableau is $x = 48$, $y = 84$, $z = 0$, $s = 0$, $t = 0$, $u = 60$, and $P = 708$.
2. The interpretation of the solution is that if 48 souvenirs of type A, 84 of type B and none of type C are made, there will be a profit of $708.
3. Since there are no negative indicators below the line, the objective function has been maximized.